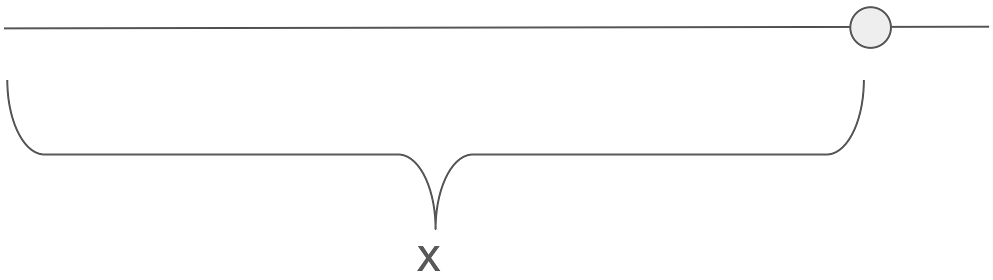
Erlang Distribution

Recap

Assume: expected number of successes produced in x seconds is $r \cdot x$

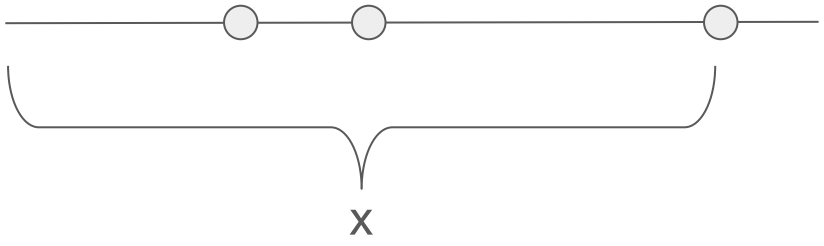
Exponential random variable X describes interval between two successes of a constant rate (Poisson) random process with success rate r per unit interval.



How to model the interval X to the k^{th} event of a constant rate process?

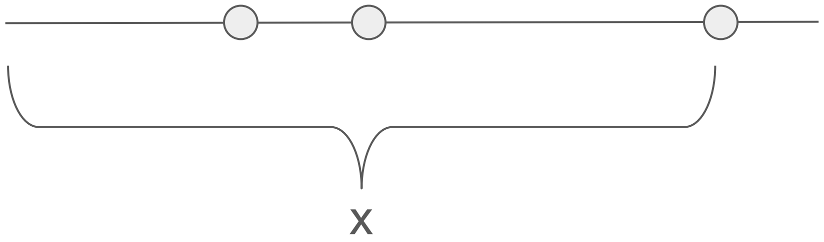
Consider

P(X>x) = There have been less than k events in time x



How to model the interval X to the k^{th} event of a constant rate process?

Consider



P(X>x) = There have been less than k events in time x

= P(Nx = 0) + P(Nx=1) +...+P(Nx=k-1)

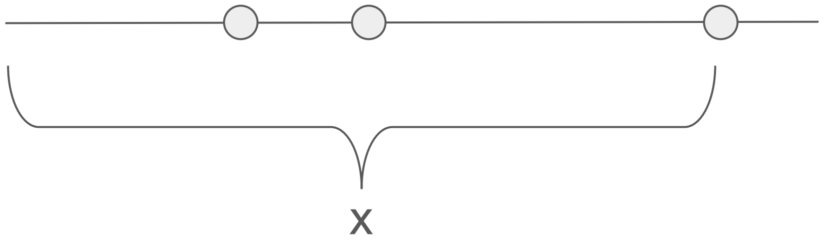
How to model the interval X to the k^{th} event of a constant rate process?

Consider

$P(X>x)$ = There have been less than k events in time x

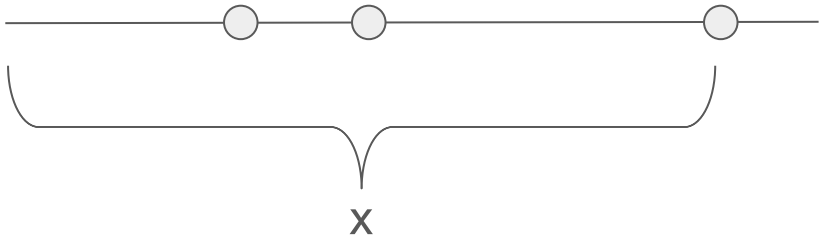
$= P(Nx = 0) + P(Nx=1) + ... + P(Nx=k-1)$

Where $P(Nx = m) = \dfrac{e^{-rx} (rx)^m}{m!}$



How to model the interval X to the k^{th} event of a constant rate process?

Consider



P(X>x) = There have been less than k events in time x

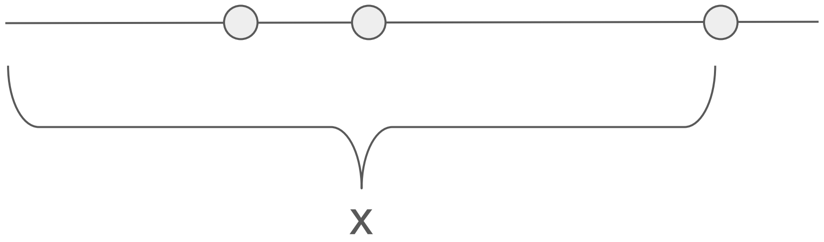
= P(Nx = 0) + P(Nx=1) +...+P(Nx=k-1)

Where P(Nx = m) = $\dfrac{e^{-rx}(rx)^m}{m!}$

$$P(X > x) = \sum_{m=0}^{k-1} \frac{e^{-rx}(rx)^m}{m!} = 1 - F(x)$$

How to model the interval X to the kth event of a constant rate process?

$$P(X > x) = \sum_{m=0}^{k-1} \frac{e^{-rx}(rx)^m}{m!} = 1 - F(x)$$



Differentiating F(x) we find that all terms in the sum except the last one cancel each other:

$$f(x) = \frac{r^k x^{k-1} e^{-rx}}{(k-1)!} \quad \text{for } x > 0 \text{ and } k = 1,2,3,\ldots$$

Can we generalize it further?

What if k was not a non-negative integer?

The Gamma distribution

What if k was not a non-negative integer?

The factorial can be generalized with the Gamma function:

$$f(x) = \frac{r^k x^{k-1} e^{-rx}}{\Gamma(k)}, \text{ for } x > 0$$

The Gamma distribution

What if k was not a non-negative integer?

The factorial can be generalized with the Gamma function:

$$f(x) = \frac{r^k x^{k-1} e^{-rx}}{\Gamma(k)}, \text{ for } x > 0$$

Remember

$$\int_0^{+\infty} f(x)dx = 1$$

We can use that to find an expression for Γ(k)

The Gamma function

$$\int_0^{+\infty} f(x)dx = 1 \qquad f(x) = \frac{r^k x^{k-1} e^{-rx}}{\Gamma(k)}, \text{ for } x > 0$$

$$\Gamma(k) = \int_0^{+\infty} r^k x^{k-1} e^{-rx} dx = \int_0^{+\infty} y^{k-1} e^{-y} dy$$

Note that for integer k one gets:
$\Gamma(k) = (k-1)!$

The Gamma distribution

If X is an Erlang (or more generally Gamma) random variable with parameters r and k,

$$\mu = E(X) = k/r \quad \text{and } \sigma^2 = V(X) = k/r^2$$

Compare with:

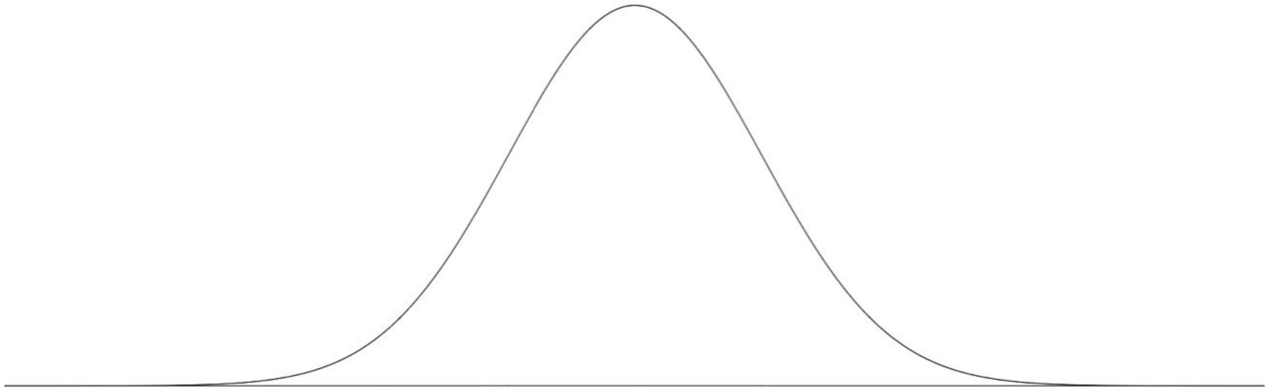
	Exponential	Negative binomial
Mean	$\mu = E(X) = 1/r$	$\mu = E(X) = k/p$
Variance	$\sigma^2 = V(X) = 1/r^2$	$\sigma^2 = V(X) = k(1-p) / p^2$

Matlab Exercise

1. Generate a sample of 100,000 variables with Exponential distribution with r =0.1
2. Generate a sample of 100,000 variables with “Harry Potter” Gamma distribution with r =0.1 and k=9 and ¾ (9.75)
3. Generate a sample of 100,000 variables with the Gamma distribution with r=0.1 and k=1.

- Calculate mean and standard deviation and compare them to 1/r (Exp) and k/r (Gamma)
- Plot semilog-y plots of PDFs **and CCDFs.**
- Hint: read the help page (better yet documentation webpage) for random and scroll down to find which parameters to use: one of their parameters is different than r

See anything interesting?



Normal distribution

Paranormal distribution

Normal/Gaussian Distribution

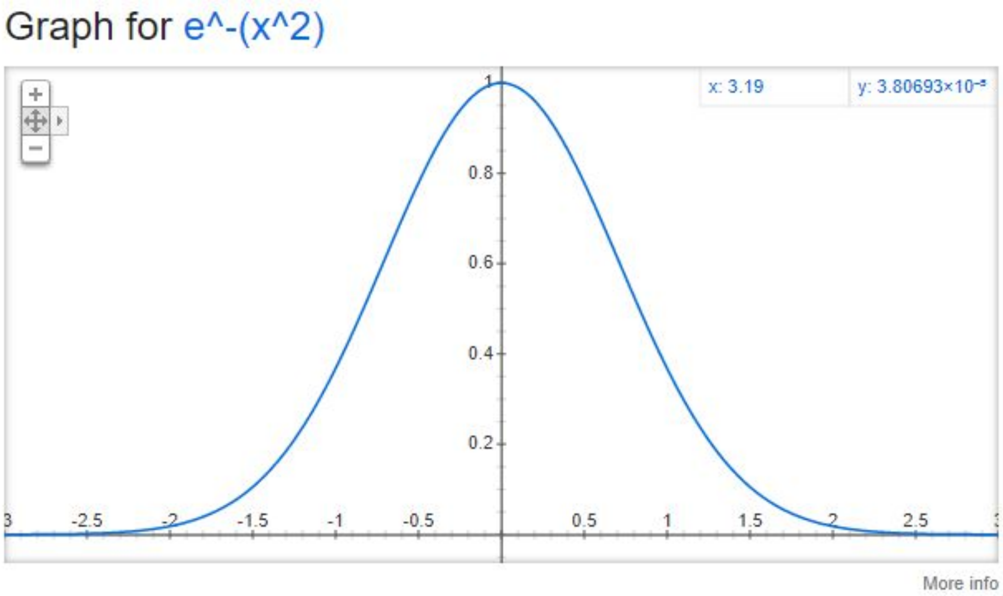
Gaussian Distribution

$$f(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

$$-\infty < x < \infty$$

Carl Friedrich Gauss (1777 –1855)
German mathematician

Gaussian Distribution

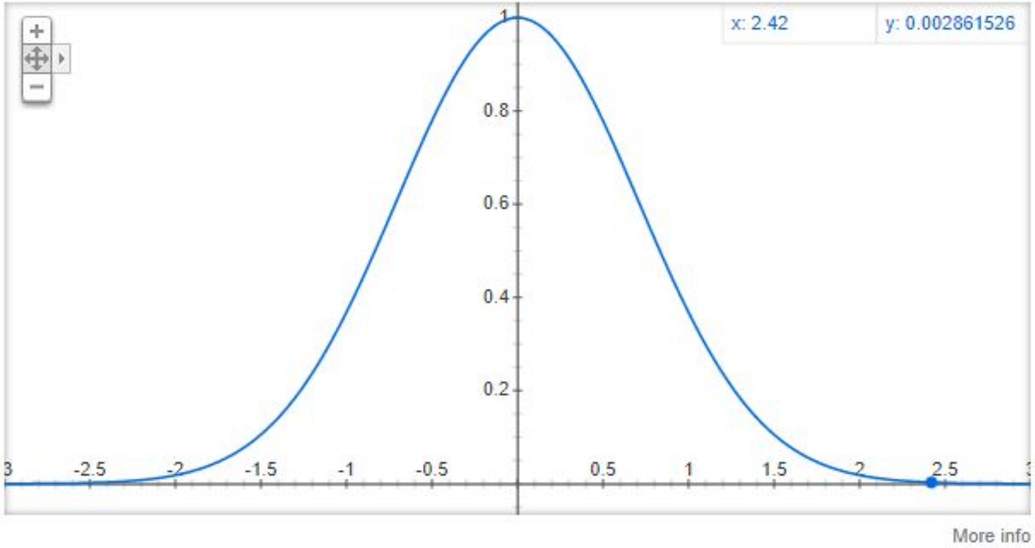


On Google, search for:
e^-(x^2) from -3 to 3

$$f(x) = e^{-x^2}$$

Gaussian Distribution: What do the individual parts do?

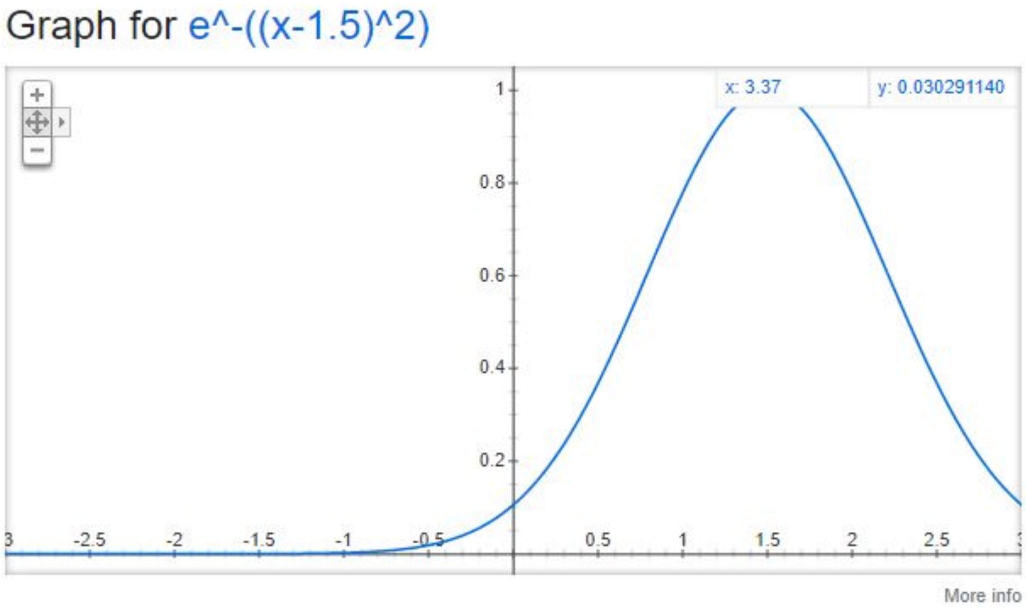
Graph for e^-((x-0)^2)



On Google, search for:
e^-((x-0)^2) from -3 to 3

$$f(x) = e^{-(x-\mu)^2}$$

Gaussian Distribution: What do the individual parts do?

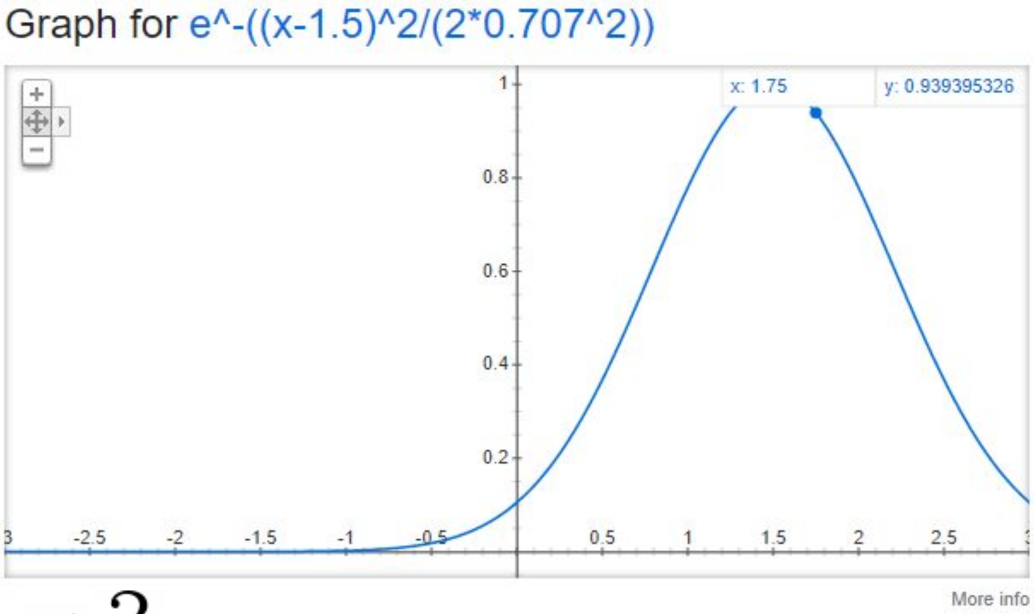


On Google, search for:
e^-((x-1.5)^2) from -3 to 3

$$f(x) = e^{-(x-\mu)^2}$$

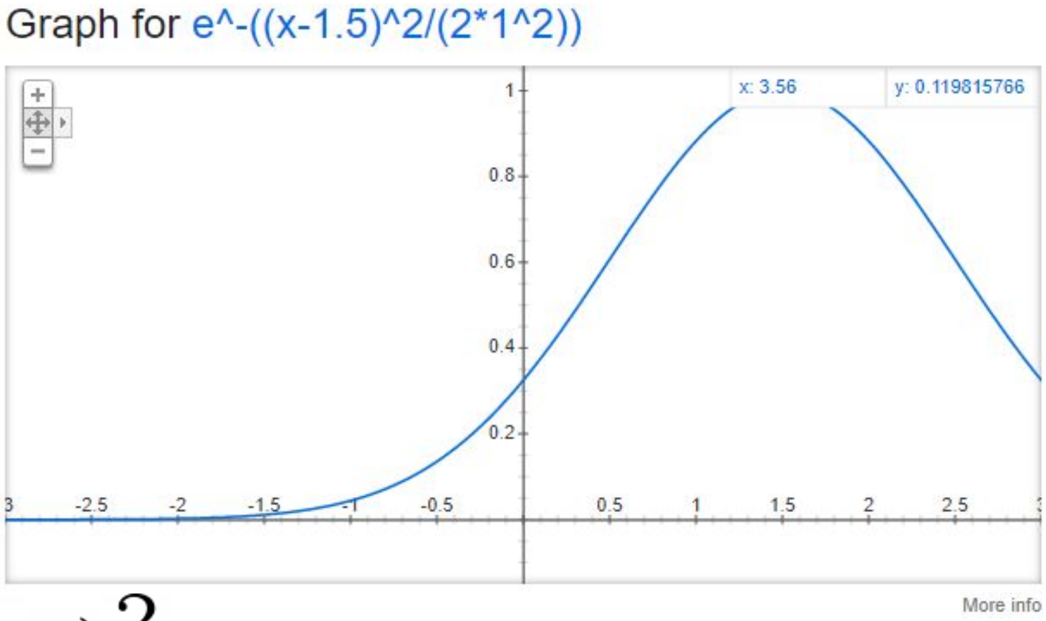
Gaussian Distribution: What do the individual parts do?

On Google, search for:
e^-((x-1.5)^2/(2(0.707)^2)) from -3 to 3



$$f(x) = e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

Gaussian Distribution: What do the individual parts do?

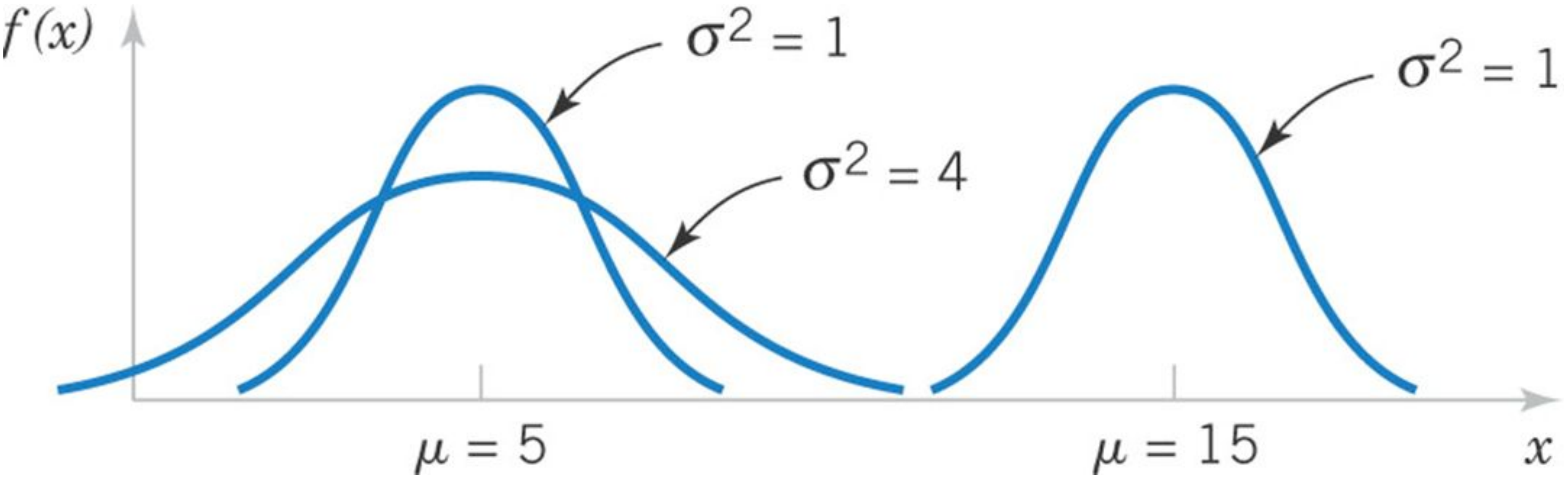


On Google, search for:
e^-((x-1.5)^2/(2(1)^2)) from -3 to 3

$$f(x) = e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

The only part we are missing now is the normalizing constant

Gaussian Distribution: What do the individual parts do?



The location and spread of the normal are independently determined by mean (μ) and standard deviation (σ)

Why is the Gaussian distribution so important?

Any sum of many independent random variables can be approximated with a Gaussian.

The distribution of sample means approximates a normal distribution as the sample size gets larger, regardless of the population's distribution (a.k.a. Central Limit Theorem)

Standard Normal Distribution

- A normal (Gaussian) random variable with

$$\mu = 0 \text{ and } \sigma^2 = 1$$

  is called a standard normal random variable and is denoted as Z.

- The cumulative distribution function of a standard normal random variable is denoted as:

$$\Phi(z) = P(Z \leq z)$$

- You will be given a table of values for this function in exams!

Standardizing

If X is a normal random variable with $E(X) = \mu$ and $V(X) = \sigma^2$, the random variable

$$Z = \frac{X - \mu}{\sigma} \qquad (4\text{-}10)$$

is a normal random variable with $E(Z) = 0$ and $V(Z) = 1$. That is, Z is a standard normal random variable.

Suppose X is a normal random variable with mean μ and variance σ^2.

Then, $P(X \le x) = P\left(\frac{X-\mu}{\sigma} \le \frac{x-\mu}{\sigma}\right) = P(Z \le z) \qquad (4\text{-}11)$

where Z is a standard normal random variable, and

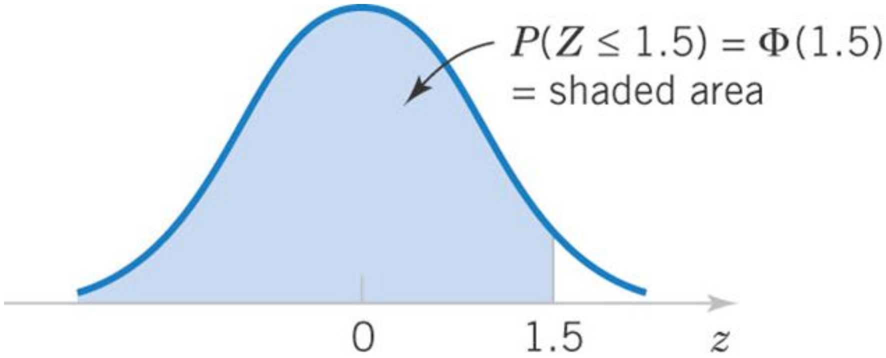
$z = \frac{(x-\mu)}{\sigma}$ is the z-value obtainedby standardizing x.

The probability is obtained by using Appendix Table III

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.500000	0.503989	0.507978	0.511967	0.515953	0.519939	0.532922	0.527903	0.531881	0.535856
0.1	0.539828	0.543795	0.547758	0.551717	0.555760	0.559618	0.563559	0.567495	0.571424	0.575345
0.2	0.579260	0.583166	0.587064	0.590954	0.594835	0.598706	0.602568	0.606420	0.610261	0.614092
0.3	0.617911	0.621719	0.625516	0.629300	0.633072	0.636831	0.640576	0.644309	0.648027	0.651732
0.4	0.655422	0.659097	0.662757	0.666402	0.670031	0.673645	0.677242	0.680822	0.684386	0.687933
0.5	0.691462	0.694974	0.698468	0.701944	0.705401	0.708840	0.712260	0.715661	0.719043	0.722405
0.6	0.725747	0.729069	0.732371	0.735653	0.738914	0.742154	0.745373	0.748571	0.751748	0.754903
0.7	0.758036	0.761148	0.764238	0.767305	0.770350	0.773373	0.776373	0.779350	0.782305	0.785236
0.8	0.788145	0.791030	0.793892	0.796731	0.799546	0.802338	0.805106	0.807850	0.810570	0.813267
0.9	0.815940	0.818589	0.821214	0.823815	0.826391	0.828944	0.831472	0.833977	0.836457	0.838913
1.0	0.841345	0.843752	0.846136	0.848495	0.850830	0.853141	0.855428	0.857690	0.859929	0.862143
1.1	0.864334	0.866500	0.868643	0.870762	0.872857	0.874928	0.876976	0.878999	0.881000	0.882977
1.2	0.884930	0.886860	0.888767	0.890651	0.892512	0.894350	0.896165	0.897958	0.899727	0.901475
1.3	0.903199	0.904902	0.906582	0.908241	0.909877	0.911492	0.913085	0.914657	0.916207	0.917736
1.4	0.919243	0.920730	0.922196	0.923641	0.925066	0.926471	0.927855	0.929219	0.930563	0.931888
1.5	0.933193	0.934478	0.935744	0.936992	0.938220	0.939429	0.940620	0.941792	0.942947	0.944083
1.6	0.945201	0.946301	0.947384	0.948449	0.949497	0.950529	0.951543	0.952540	0.953521	0.954486
1.7	0.955435	0.956367	0.957284	0.958185	0.959071	0.959941	0.960796	0.961636	0.962462	0.963273
1.8	0.964070	0.964852	0.965621	0.966375	0.967116	0.967843	0.968557	0.969258	0.969946	0.970621
1.9	0.971283	0.971933	0.972571	0.973197	0.973810	0.974412	0.975002	0.975581	0.976148	0.976705
2.0	0.977250	0.977784	0.978308	0.978822	0.979325	0.979818	0.980301	0.980774	0.981237	0.981691
2.1	0.982136	0.982571	0.982997	0.983414	0.983823	0.984222	0.984614	0.984997	0.985371	0.985738
2.2	0.986097	0.986447	0.986791	0.987126	0.987455	0.987776	0.988089	0.988396	0.988696	0.988989
2.3	0.989276	0.989556	0.989830	0.990097	0.990358	0.990613	0.990863	0.991106	0.991344	0.991576
2.4	0.991802	0.992024	0.992240	0.992451	0.992656	0.992857	0.993053	0.993244	0.993431	0.993613
2.5	0.993790	0.993963	0.994132	0.994297	0.994457	0.994614	0.994766	0.994915	0.995060	0.995201
2.6	0.995339	0.995473	0.995604	0.995731	0.995855	0.995975	0.996093	0.996207	0.996319	0.996427
2.7	0.996533	0.996636	0.996736	0.996833	0.996928	0.997020	0.997110	0.997197	0.997282	0.997365
2.8	0.997445	0.997523	0.997599	0.997673	0.997744	0.997814	0.997882	0.997948	0.998012	0.998074
2.9	0.998134	0.998193	0.998250	0.998305	0.998359	0.998411	0.998462	0.998511	0.998559	0.998605
3.0	0.998650	0.998694	0.998736	0.998777	0.998817	0.998856	0.998893	0.998930	0.998965	0.998999
3.1	0.999032	0.999065	0.999096	0.999126	0.999155	0.999184	0.999211	0.999238	0.999264	0.999289
3.2	0.999313	0.999336	0.999359	0.999381	0.999402	0.999423	0.999443	0.999462	0.999481	0.999499
3.3	0.999517	0.999533	0.999550	0.999566	0.999581	0.999596	0.999610	0.999624	0.999638	0.999650
3.4	0.999663	0.999675	0.999687	0.999698	0.999709	0.999720	0.999730	0.999740	0.999749	0.999758
3.5	0.999767	0.999776	0.999784	0.999792	0.999800	0.999807	0.999815	0.999821	0.999828	0.999835
3.6	0.999841	0.999847	0.999853	0.999858	0.999864	0.999869	0.999874	0.999879	0.999883	0.999888
3.7	0.999892	0.999896	0.999900	0.999904	0.999908	0.999912	0.999915	0.999918	0.999922	0.999925
3.8	0.999928	0.999931	0.999933	0.999936	0.999938	0.999941	0.999943	0.999946	0.999948	0.999950
3.9	0.999952	0.999954	0.999956	0.999958	0.999959	0.999961	0.999963	0.999964	0.999966	0.999967

How to use the table?

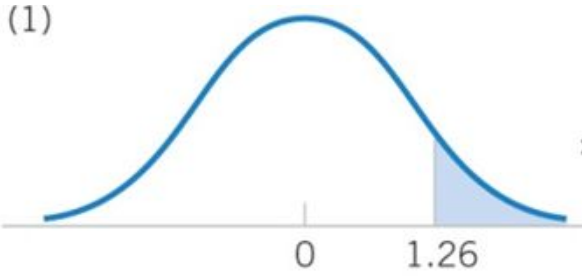
Find $P(Z \le 1.50)$ Answer: 0.93319



z	0.00	0.01	0.02	0.03
0	0.50000	0.50399	0.50398	0.51197
⋮		⋮		
1.5	0.93319	0.93448	0.93574	0.93699

Find $P(Z \le 1.53)$

More complex examples



$P(Z > 1.26)$

$= 1 - P(Z < 1.26)$

$= 1 - \Phi(1.26)$

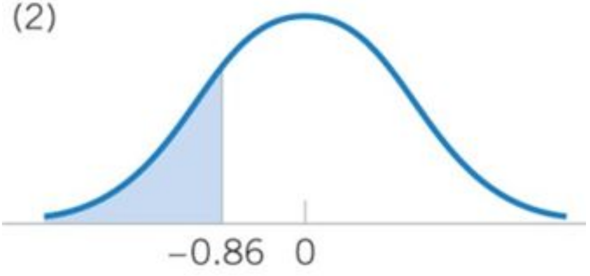
More complex examples

$P(Z < -0.86)$

$= P(Z > 0.86)$

$= 1- P(Z < 0.86)$

$= 1- \Phi(0.86)$



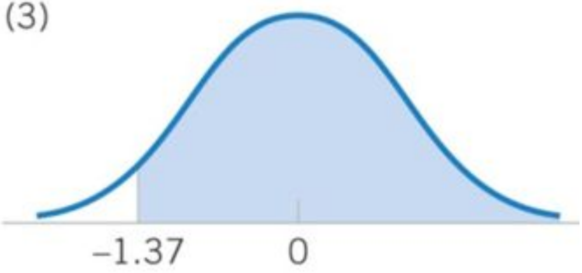
More complex examples

$P(Z > -1.37)$

$= P(Z < 1.37)$

$= \Phi(1.37)$

(3)



More complex examples

$P(-1.25 < Z < 0.37)$

$= P(Z < 0.37) - P(Z < -1.25)$

$= P(Z < 0.37) - (1 - P(Z < 1.25))$